Math 524 Exam 4 Solutions

For each of the following vector spaces V and linear operators L:

1. Find all eigenvalues.
2. Find a basis for each eigenspace.
3. Determine all algebraic and geometric multiplicities.
4. Is the operator diagonalizable?

A. $V = \mathbb{R}^3$, $L(x) = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & -1 \\ -2 & 4 & 5 \end{bmatrix} x$

We calculate the characteristic polynomial: $\begin{vmatrix} \lambda-3 & 0 & 0 \\ -1 & \lambda-1 & 1 \\ 2 & -4 & \lambda-5 \end{vmatrix} = (\lambda-3)^3$. Hence 3 is the only eigenvalue, of algebraic multiplicity 3 (i.e. $m_a(3) = 3$). To find its eigenspace, we seek the nullspace of $3I - A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 1 \\ 2 & -4 & -2 \end{bmatrix}$. The reduced row echelon form of this matrix is $\begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; hence its nullspace has x_2, x_3 free, and $x_1 = 2x_2 + x_3$. Thus a basis for this nullspace (hence for the eigenspace E_3) is $\{(2,1,0)^T, (1,0,1)^T\}$. The geometric multiplicity of eigenvalue 3 is therefore two (i.e. $m_g(3) = 2$), and therefore the operator L is *NOT* diagonalizable, since it is deficient.

B. $V = M_{2,2}(\mathbb{R})$, the set of all 2×2 real matrices. We have $V = W_1 + W_2$, an internal direct sum, where $W_1 = \{A : A = A^T\}$ is the subspace of symmetric matrices, and $W_2 = \{A : A = -A^T\}$ is the subspace of skew-symmetric matrices. L is the operator that projects from V to W_1.

Thoughtful solution: L fixes W_1 (i.e. $L(w) = 1w$ for $w \in W_1$), so $W_1 \subseteq E_1$, and hence $m_g(1) \geq 3$, since W_1 is 3 dimensional. Also, L kills W_2 (i.e. $L(w) = 0w$ for $w \in W_2$), so $W_2 \subseteq E_0$, and hence $m_g(0) \geq 1$, since W_2 is 1 dimensional. But V is 4 dimensional, so $4 \geq m_a(1)+m_a(0) \geq m_g(1)+m_g(0) \geq 4$, and all the inequalities are equalities (i.e. $m_a(1) = m_g(1) = 3, m_a(0) = m_g(0) = 1$),there are no other eigenvalues besides $\{1, 0\}$, and L is diagonalizable. A basis for E_1 is any basis for W_1, such as $\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$; similarly, a basis for E_0 is any basis for W_2, such as $\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}$.

Mechanical solution: Using the standard basis $S = \{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\}$, we can write $[L]_S = \begin{bmatrix}1&0&0&0\\0&1/2&1/2&0\\0&1/2&1/2&0\\0&0&0&1\end{bmatrix}$. (note that $\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right] = \left[\begin{smallmatrix}0&0.5\\0.5&0\end{smallmatrix}\right] + \left[\begin{smallmatrix}0&0.5\\-0.5&0\end{smallmatrix}\right]$, so L projects $\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]$ onto $\left[\begin{smallmatrix}0&0.5\\0.5&0\end{smallmatrix}\right]$). We calculate the characteristic polynomial: $\begin{vmatrix}\lambda-1&0&0&0\\0&\lambda-1/2&-1/2&0\\0&-1/2&\lambda-1/2&0\\0&0&0&\lambda-1\end{vmatrix} = \lambda(\lambda-1)^3$. Hence the eigenvalues are 0, with algebraic multiplicity 1 (i.e. $m_a(0) = 1$), and 1, with algebraic multiplicity 3 (i.e. $m_a(1) = 3$). To find the eigenspace of $\lambda = 0$, we seek the nullspace of $0I - [L]_S$, which has r.r.e. form $\begin{bmatrix}1&0&0&0\\0&1&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$. This is seen to be one-dimensional, hence $m_g(0) = 1$. A basis for E_0 has S-representation $\{[0,1,-1,0]^T\}$; hence is $\{\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]\}$. To find the eigenspace of $\lambda = 1$, we seek the nullspace of $1I - [L]_S$, which has r.r.e. form $\begin{bmatrix}0&1&-1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$. This is seen to be three-dimensional, hence $m_g(1) = 3$. A basis for E_1 has S-representation $\{[1,0,0,0]^T, [0,1,1,0]^T, [0,0,0,1]^T\}$; hence is $\{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\}$). Because all the geometric multiplicities equal the respective arithmetic multiplicities, L is diagonalizable.

C. $V = C^1(\mathbb{R})$, the set of continuously differentiable functions on the real line, $L(f) = t\frac{df}{dt}$

We try to solve the differential equation $L(f) = t\frac{df}{dt} = \lambda f$. We use the standard trick for separable differential equations: $\frac{df}{f} = \lambda\frac{dt}{t}$, which we integrate to get $\ln f = \lambda \ln t + c$, hence $f = e^c t^\lambda$. Hence there is a solution for *every* eigenvalue λ; a basis for E_λ is $\{t^\lambda\}$. Note that each eigenspace is at most one-dimensional since the differential equation is first-order. Every geometric multiplicity is therefore 1 (i.e. $m_g(\lambda) = 1$ for every $\lambda \in \mathbb{R}$).

The remaining parts of this answer are quite subtle, so you will not be marked off should your answer differ. The notion of "algebraic multiplicity" has not been defined for this (or any infinite-dimensional) vector space, so this question is moot; however this operator is not diagonalizable (whatever that means in this context), since the set of all eigenvectors is not a basis. For example, $f(t) = \sin t$ is in V, but is not a linear combination of the eigenvectors $\{t^\lambda\}$.